Lecture contents

- **Macroscopic Electrodynamics -2**
  - **Skin effect**
  - **Boundary conditions, wave propagation through interface**
  - **Wave packet, group velocity, dispersion**
  - **Circuits**

Medium with losses

- After some long algebra:

$$\alpha = \omega\sqrt{\varepsilon\mu}\left\{\frac{1}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}-1\right]\right\}^{1/2} \geq 0$$

$$\beta = \omega\sqrt{\varepsilon\mu}\left\{\frac{1}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}+1\right]\right\}^{1/2} > 0$$

- The impedance becomes complex:

$$\eta = \frac{E}{H} = \sqrt{\frac{\mu}{\tilde{\varepsilon}}} = \frac{i\omega\mu}{\gamma} = \frac{\omega\mu}{\alpha^2+\beta^2}(\beta + i\alpha)$$

For good dielectrics: $\dfrac{\varepsilon_2}{\varepsilon_1} = \dfrac{\sigma}{\omega\varepsilon} \ll 1$

$$\alpha = \frac{\omega}{2}\sqrt{\varepsilon\mu}\,\frac{\sigma}{\omega\varepsilon} \qquad v_p = \frac{1}{\sqrt{\mu\varepsilon}}$$

$$\beta = \omega\sqrt{\varepsilon\mu}\left[1+\frac{1}{8}\left(\frac{\sigma}{\omega\varepsilon}\right)^2\right]$$

$$\eta = \sqrt{\frac{\mu}{\varepsilon}}\left\{\left[1-\frac{3}{8}\left(\frac{\sigma}{\omega\varepsilon}\right)^2\right]+i\,\frac{1}{2}\left(\frac{\sigma}{\omega\varepsilon}\right)\right\}$$

For good conductors: $\dfrac{\varepsilon_2}{\varepsilon_1} = \dfrac{\sigma}{\omega\varepsilon} \gg 1$

$$\alpha = \beta = \sqrt{\frac{\omega\mu\sigma}{2}}$$

$$v_p = \sqrt{\frac{2\omega}{\mu\sigma}}$$

$$\eta = \sqrt{\frac{\omega\mu}{2\sigma}} + i\sqrt{\frac{\omega\mu}{2\sigma}}$$

Skin depth

- In a good conductor (metal) the attenuation of the wave is determined by attenuation constant *a* :

$$\alpha = \sqrt{\frac{\omega\mu\sigma}{2}}$$

- The ***skin depth*** of material is the depth to which a uniform plane wave can penetrate before it is attenuated by a factor of *1/e*. For planar surfaces, skin depth is given as:

$$\delta_{skin} = \frac{1}{\alpha} = \sqrt{\frac{2}{\omega\mu\sigma}}$$

- Penetration depends on frequency ($f^{-1/2}$), conductivity and permeability.
- The skin effect is the result of induction: time-varying magnetic field is accompanied by a time varying electric field → time varying current →secondary fields
- Skin-effect implies dissipation of the wave power by the current

Skin effect

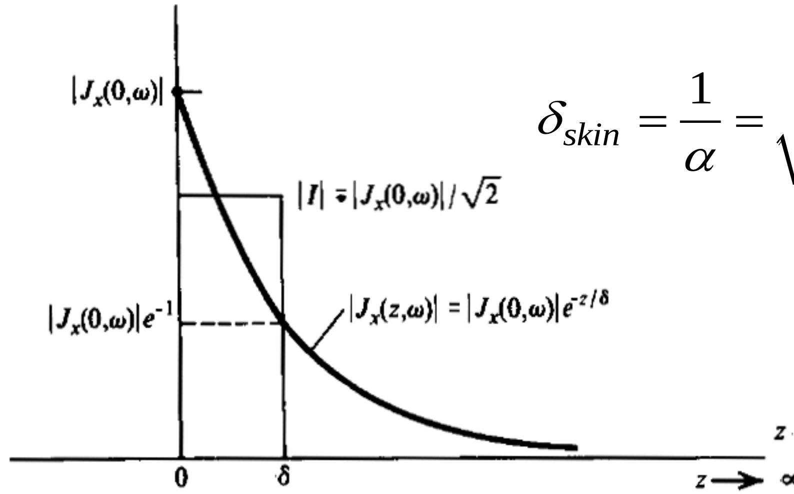
$$J_x(z,\omega) = J_x(0,\omega)e^{-i\frac{z}{\delta}}e^{-\frac{z}{\delta}}\ ,\ z>0$$

- Another result of skin effect is non-uniform distribution of current in conductors at high frequencies
- Current density drops into the metal at high frequencies due to screening of the EM field by the induced current
- Usually in engineering disciplines it is modeled as if all current flowed in δ-thick outer layer of conductor
- As a result, resistance of a thick conductor increase with frequency.

$$R \approx \frac{l}{2\pi a\sigma\delta}$$

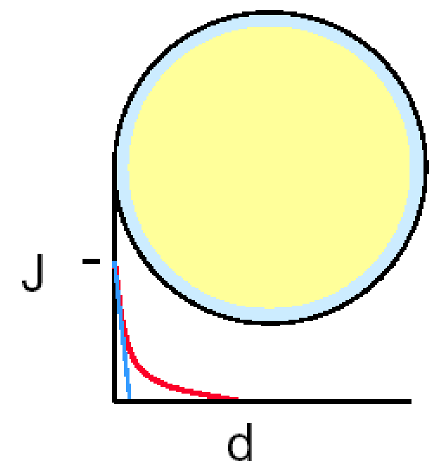
- If a wire radius $a >> \delta$,
- Skin layer thickness in Cu:

frequency	d
60 Hz	8.57 mm
10 kHz	0.66 mm
100 kHz	0.21 mm
1 MHz	66 μm
10 MHz	21 μm

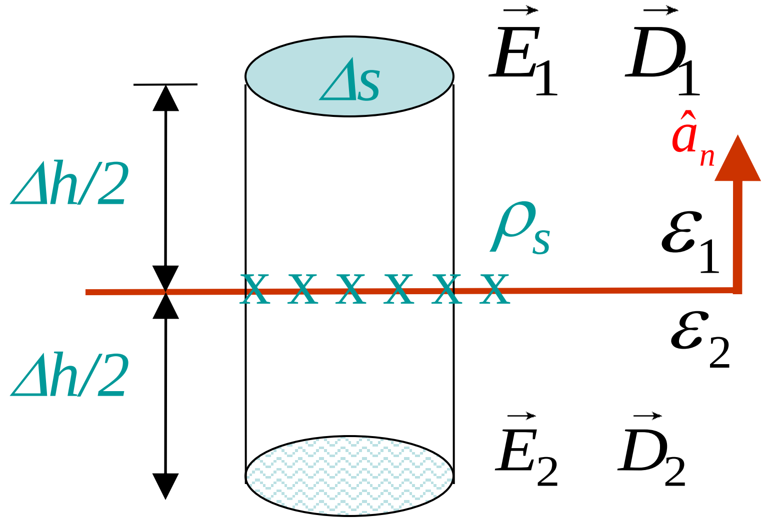


$$\delta_{skin} = \frac{1}{\alpha} = \sqrt{\frac{2}{\omega\mu\sigma}}$$

From Neff, 1991



Example of Boundary Condition: Normal Component of *D*



- Consider electric field normal to the boundary
- Consider a cylinder with cross-sectional area Δs and height Δh lying half in medium 1 and half in medium 2:
- Applying Gauss's law to the cylinder

$$\oint_S \vec{D} \cdot d\vec{s} = \int_V q_f \, dv$$

$$\int_{top} \vec{D} \cdot d\vec{s} + \int_{bottom} \vec{D} \cdot d\vec{s} + \int_{side} \vec{D} \cdot d\vec{s} \quad (\to 0)$$

$$= D_{1n} \Delta s - D_{2n} \Delta s$$

$$q_f \Delta s$$

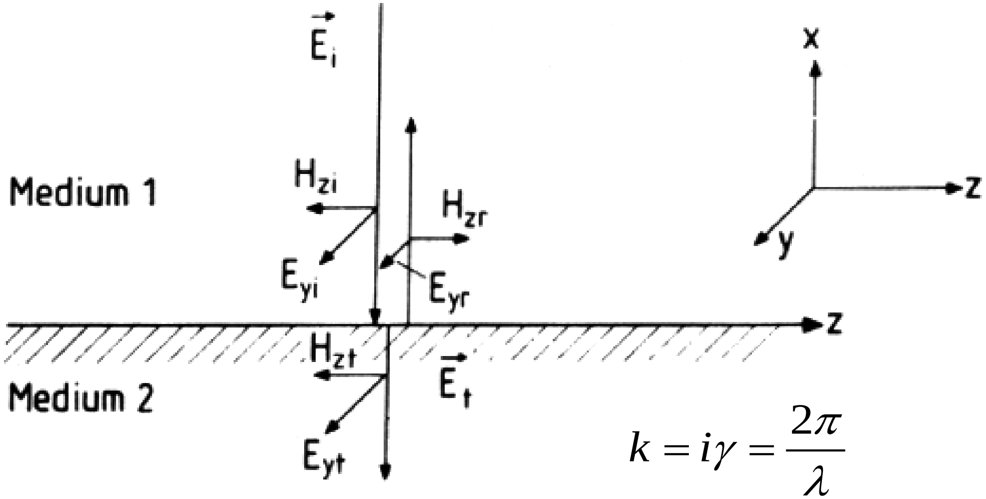
- The boundary condition is $D_{1n} - D_{2n} = \rho_s$

- If there is no surface charge

$$\boxed{D_{1n} = D_{2n}} \quad \text{or} \quad \boxed{\varepsilon_1 E_{1n} = \varepsilon_2 E_{2n}}$$

For *non-conducting* materials, $\rho_s = 0$

Wave propagation through the interface: boundary conditions and normal incidence (low absorption)



$$k = i\gamma = \frac{2\pi}{\lambda}$$

Boundary conditions result from application of Maxwell equations to the interface area:

$$\mathbf{E}_{1tan} = \mathbf{E}_{2tan}$$

$$\mathbf{H}_{1tan} = \mathbf{H}_{2tan}$$

$$\mathbf{D}_{1norm} = \mathbf{D}_{2norm}$$

$$\mathbf{B}_{1norm} = \mathbf{B}_{2norm}$$

- Plane transverse EM (TEM) waves propagating in --x direction:

$$\begin{cases} E_y = E_0 e^{i\omega t} e^{i\beta x + \alpha x} + c.c. \\ H_z = H_0 e^{i\omega t} e^{i\beta x + \alpha x} + c.c. \\ \dfrac{E_0}{H_0} = \sqrt{\dfrac{\mu}{\tilde{\varepsilon}}} = \eta \end{cases}$$

- Need to consider 3 waves: incident (i), reflected (r) and transmitted (t)

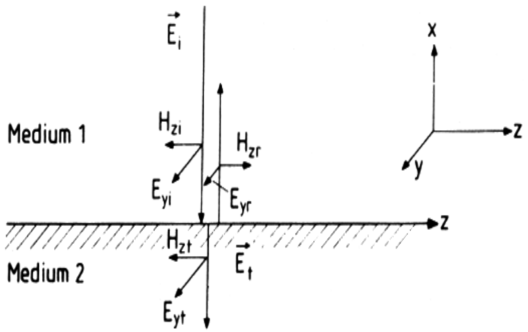
- Continuity of tangential E fields requires: $E_i + E_r = E_t$

- Continuity of tangential H fields requires: $H_i + H_r = H_t$ or $\dfrac{E_i}{\eta_1} - \dfrac{E_r}{\eta_1} = \dfrac{E_t}{\eta_2}$

equivalent result in optics (nonmagnetic): $\tilde{n}_1 E_i - \tilde{n}_1 E_r = \tilde{n}_2 E_t$ since $\dfrac{E_0}{H_0} = \sqrt{\dfrac{\mu_0}{\varepsilon_0}} \dfrac{1}{\tilde{n}}$

Wave propagation through the interface: boundary conditions and normal incidence

- Amplitude reflection and transmission for normal incidence



$$r = \frac{E_r}{E_i} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \qquad t = \frac{E_t}{E_i} = \frac{2\eta_2}{\eta_2 + \eta_1} = 1 + r \qquad \frac{H_t}{H_i} = \frac{2\eta_1}{\eta_2 + \eta_1}$$

Or from refractive indexes (notation used in optics)

$$r = \frac{E_r}{E_i} = \frac{\tilde{n}_1 - \tilde{n}_2}{\tilde{n}_1 + \tilde{n}_2} \qquad t = \frac{E_t}{E_i} = \frac{2\tilde{n}_1}{\tilde{n}_1 + \tilde{n}_2}; \qquad \frac{H_t}{H_i} = \frac{2\tilde{n}_2}{\tilde{n}_1 + \tilde{n}_2}$$

- Power or intensity reflection and transmission for normal incidence

$$R = \left| \frac{\tilde{n}_1 - \tilde{n}_2}{\tilde{n}_1 + \tilde{n}_2} \right|^2 \qquad T = \frac{\mathrm{Re}(\tilde{n}_2)}{\mathrm{Re}(\tilde{n}_1)} \left| \frac{2\tilde{n}_1}{\tilde{n}_1 + \tilde{n}_2} \right|^2 = 1 - R$$

- Dielectric functions, or refraction/extinction indexes or impedances is sufficient to describe linear EM properties of a non-magnetic medium.

Normal incidence from dielectric to perfect metal

- Example: normal incidence from dielectric to perfect metal: perfect reflection

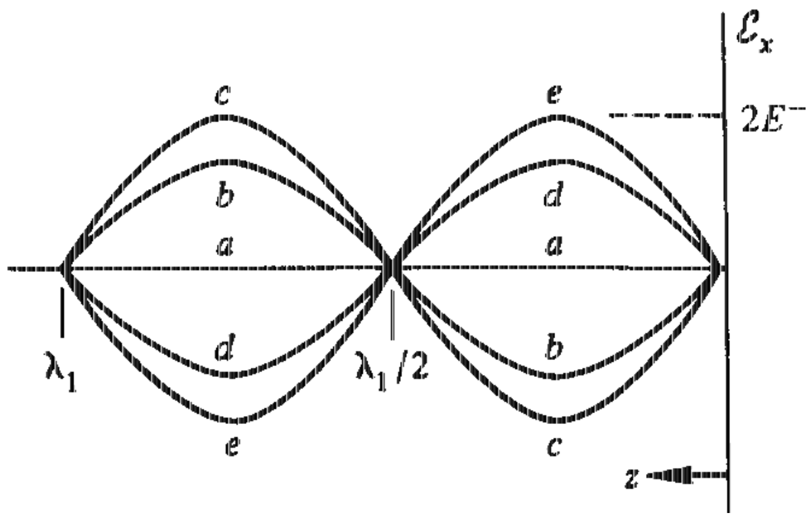
$$\eta_1 = \sqrt{\frac{\mu}{\varepsilon}}\left\{\left[1-\frac{3}{8}\left(\frac{\sigma}{\omega\varepsilon}\right)^2\right]+i\,\frac{1}{2}\left(\frac{\sigma}{\omega\varepsilon}\right)\right\} \approx \sqrt{\frac{\mu}{\varepsilon}}$$

$$\eta_2 = \sqrt{\frac{\omega\mu}{2\sigma}} + i\sqrt{\frac{\omega\mu}{2\sigma}} \approx 0$$

$$r = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \approx -1$$

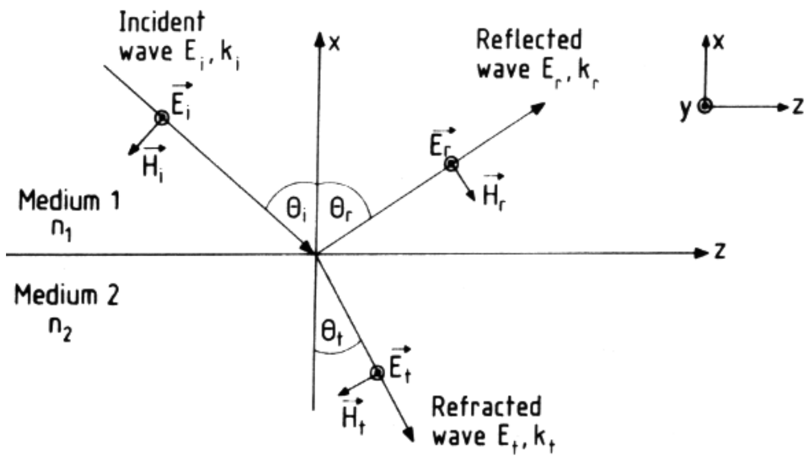
π – phase shift

- Results in a standing wave:

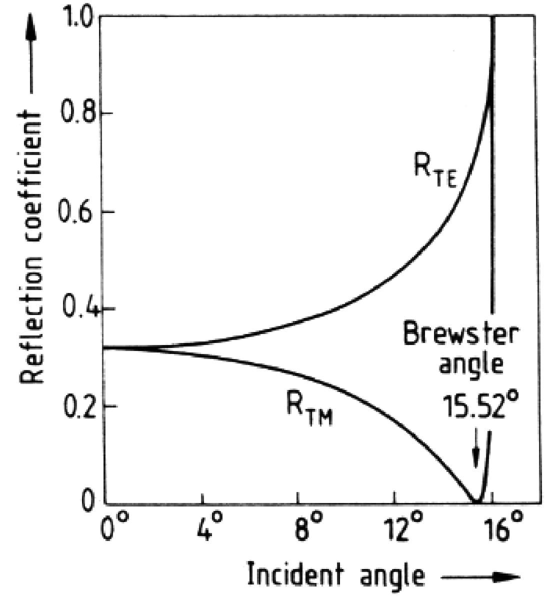


ωt	curve
0	a
$\pi/4$	b
$\pi/2$	c
$3\pi/4$	b
π	a
$5\pi/4$	d
$3\pi/2$	e
$7\pi/4$	d
2π	a

Oblique incidence: transverse electrical (TE) and transverse magnetic (TM) EM waves: refractive index



Intensity reflection coefficient from GaAs (n=3.6) to air



$$\tan\theta_{Brewster} = \frac{n_2}{n_1}$$

If incidence is not normal, reflectance will depend on polarization. Applying boundary conditions for electric field, we can obtain:

The Brewster angle is a special angle of incidence for which $r_{TM} = 0$.

For s-polarization (TE): $$r_{TE} = \frac{n_1\cos\theta_i - n_2\cos\theta_t}{n_1\cos\theta_i + n_2\cos\theta_t}$$

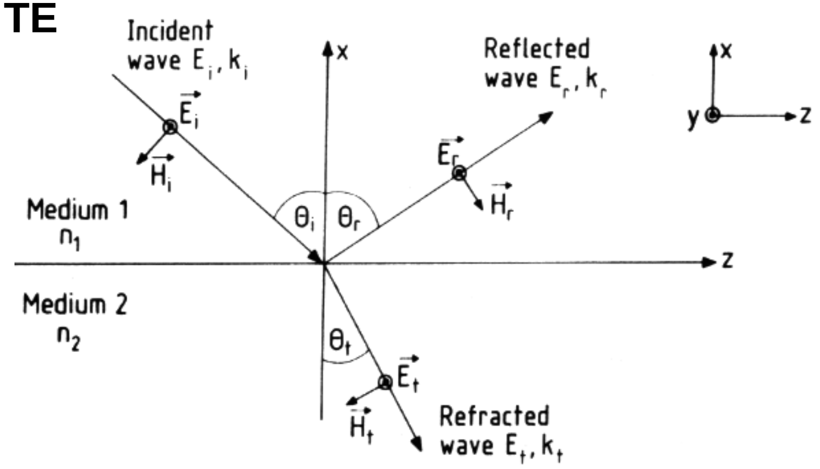
For p-polarization (TM): $$r_{TM} = \frac{n_2\cos\theta_i - n_1\cos\theta_t}{n_2\cos\theta_i + n_1\cos\theta_t}$$

$$\cos\theta_t = \sqrt{1 - \left(\frac{n_1}{n_2}\sin\theta_i\right)^2}$$

Imaginary at "Total internal reflectance"

Oblique incidence: transverse electrical (TE) and transverse magnetic (TM) EM waves: impedance

TE

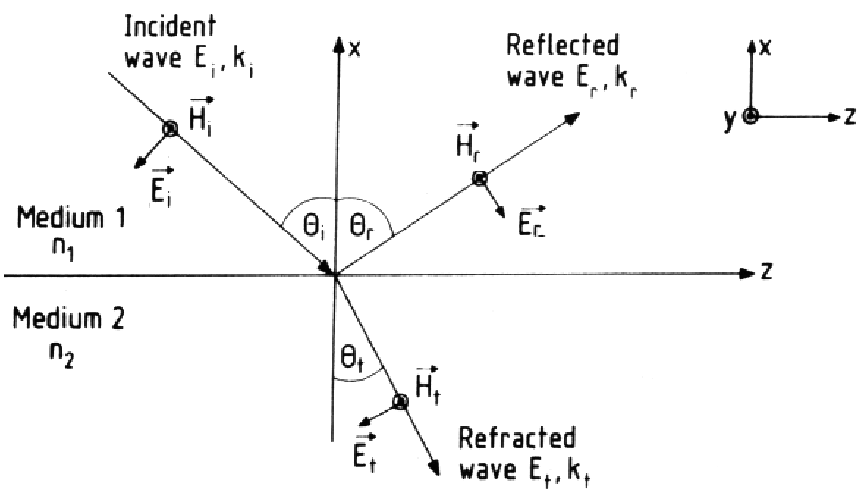


For s-polarization (TE):

$$r_{TE} = \frac{\eta_2 \cos\theta_i - \eta_1 \cos\theta_t}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t}$$

$$t_{TE} = \frac{2\eta_2 \cos\theta_i}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t}$$

TM



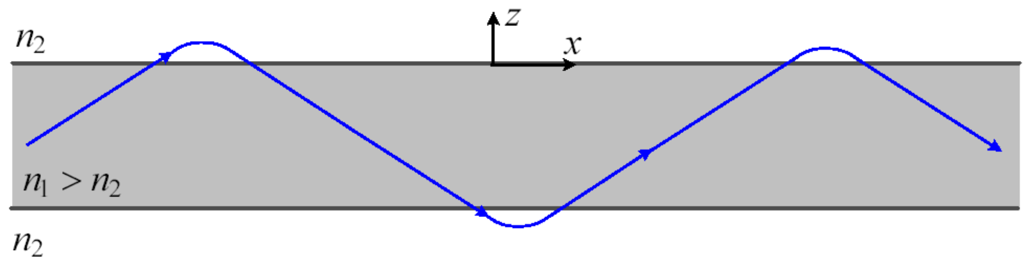
For p-polarization (TM):

$$r_{TM} = \frac{\eta_2 \cos\theta_t - \eta_1 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i}$$

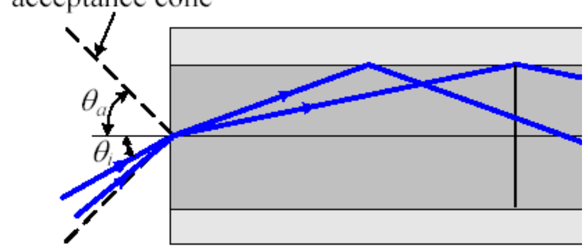
$$t_{TM} = \frac{2\eta_2 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i}$$

Example: optical waveguides

- Total internal reflection is used in dielectric waveguides /fibers



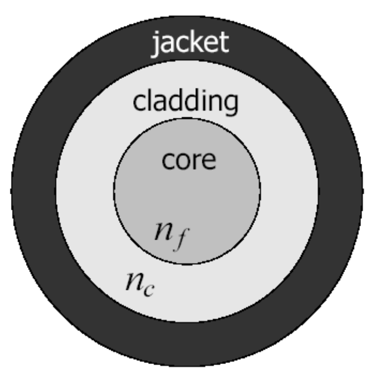
- The *acceptance angle* is the maximum angle of incidence, at which the condition for total reflection is satisfied.



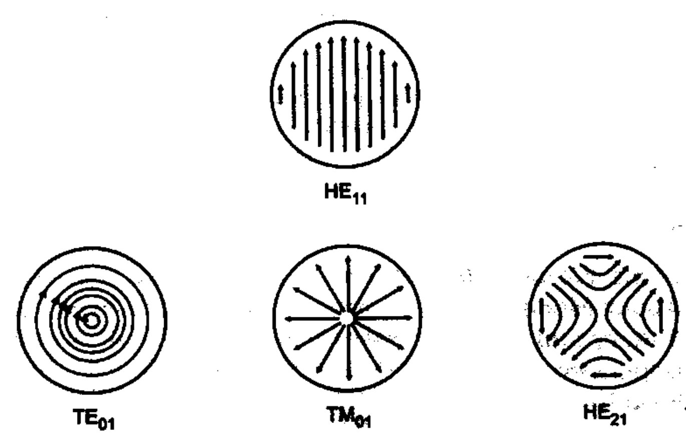
$$\theta_a = \arcsin \frac{\sqrt{n_f^2 - n_c^2}}{n_0}$$

- Solution of Maxwell equations give dispersion relationships for propagating modes, and field configurations in the modes.

- Optical fiber is another example of a waveguide

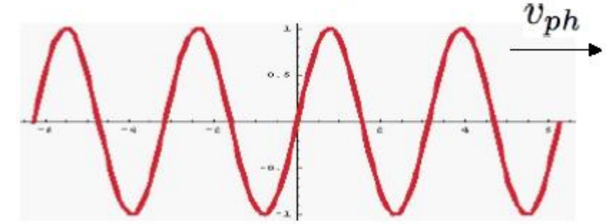


Low order modes in a fiber



Group velocity

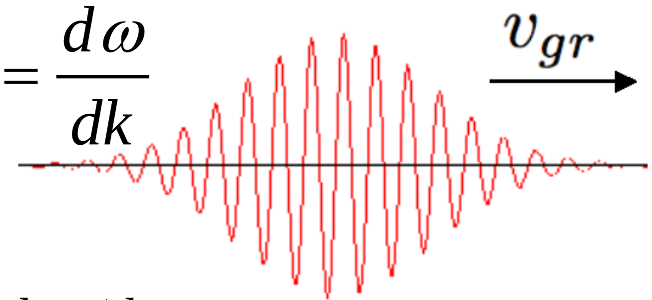
$$v_{ph} = \frac{\omega}{k}$$



- Phase velocity may depend on frequency

- If we superpose wave of several frequencies we can make a wave packet

- The velocity of the packet is the group velocity $v_g = \frac{d\omega}{dk}$



- Let'c sonsider a packet consisting of two waves with frequencies $\omega \pm \Delta\omega$ and corresponding wavevectors $k \pm \Delta k$

$$E = E_0 e^{i\left[(\omega-\Delta\omega)t-(k-\Delta k)z\right]} + E_0 e^{i\left[(\omega+\Delta\omega)t-(k+\Delta k)z\right]} = 2E_0 e^{i[\omega t - kz]} \cos\left(\Delta\omega t - \Delta k x\right)$$

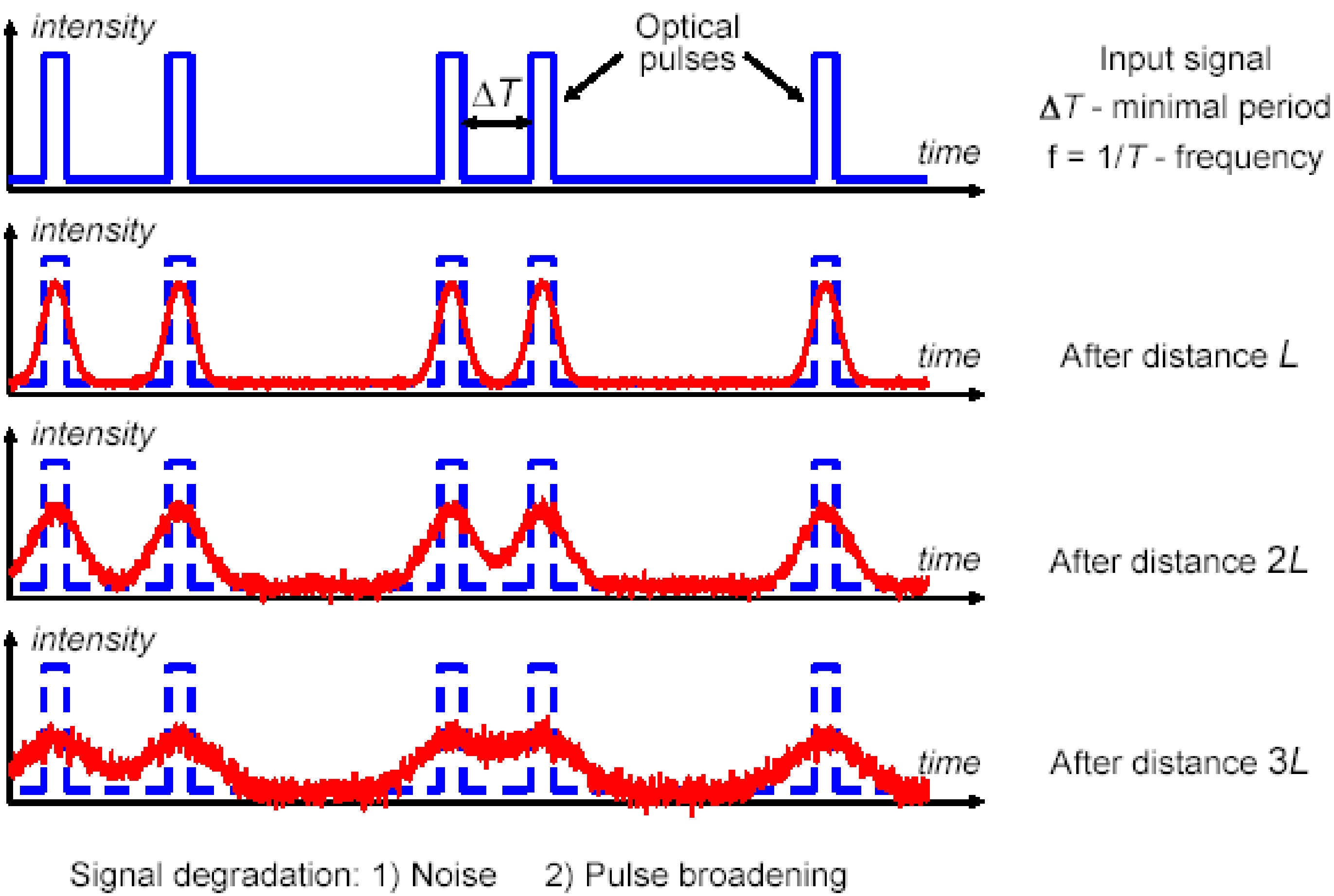
Wave propagation velocity $v_p = \frac{\omega}{k}$

Envelope propagation velocity

$$v_g = \frac{d\omega}{dk} = \left(\frac{dk}{d\omega}\right)^{-1}$$

Dispersion

- For a signal (such as a pulse) comprising a band of frequencies, different frequency components propagate with different velocities causing distortion of the signal. This phenomenon is called ***dispersion***.

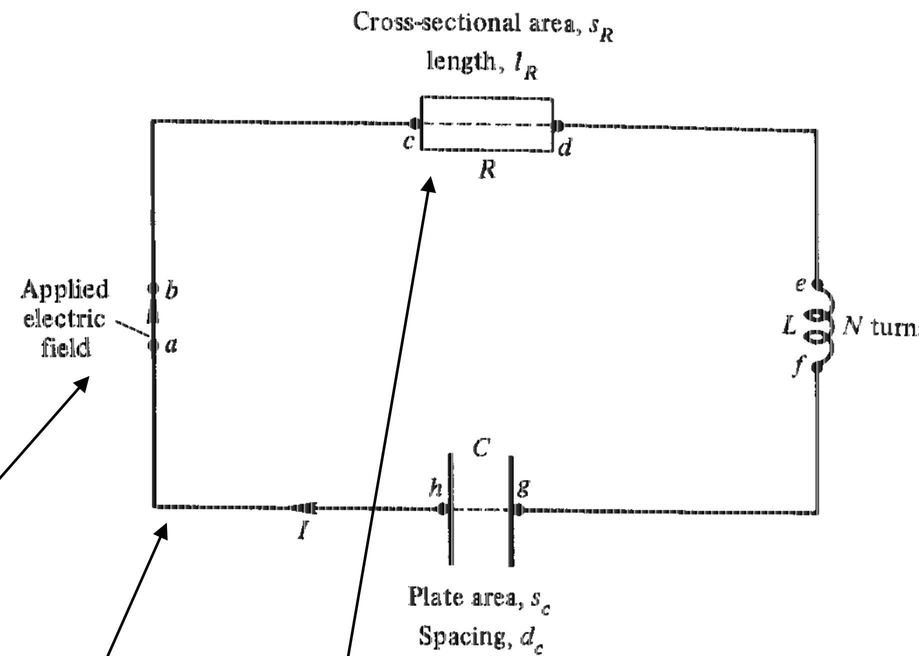


From Itskevich, 2004

Circuits

- It is informative to show that circuit theory is a very special case of electromagnetic field theory. Let's figure out where the circuit equations come from and what assumptions are made.
- Let's consider a circuit with applied voltage, resistor, capacitor and inductances and apply the Faraday's law

$$\oint_C \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\iint_S \vec{B}\cdot d\vec{s}$$



From Neff, 1991

And evaluate parts of the contour integral

- a-b Applied voltage $\int_a^b \vec{E}\cdot d\vec{l} = -V_{ba}$

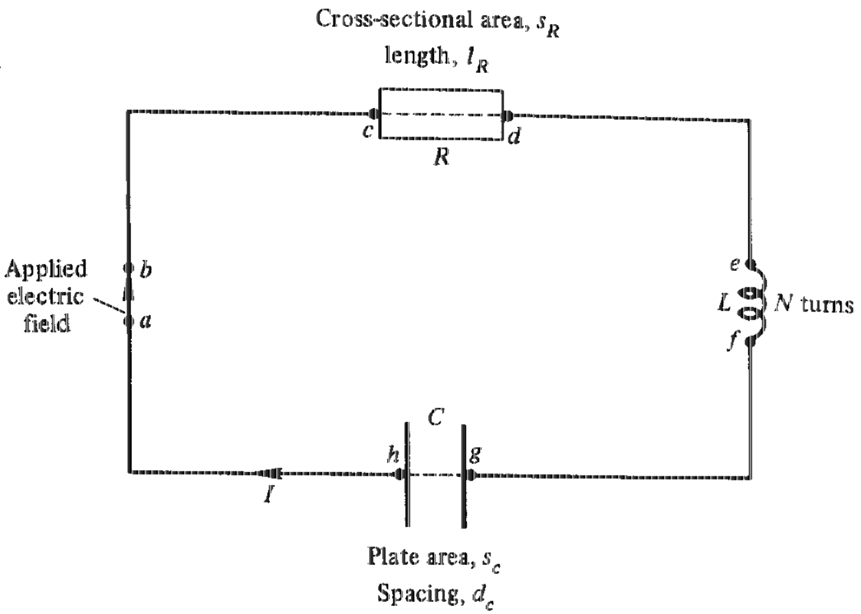
- Filamentary conductor: assume that conductivity is much higher than that of resistor → field is low $\int_{wires} \vec{E}\cdot d\vec{l} = 0$

- c-d resistor (lossy cylinder) $\int_c^d \vec{E}\cdot d\vec{l} = \int_c^d \frac{J}{\sigma}dl = \frac{J}{\sigma}L_{cd} = IR$

Circuits-2

- g-h capacitance C

$$\int_g^h \vec{E}\cdot d\vec{l} = V_{gh} = \frac{q}{C} = \frac{1}{C}\int I dt + \frac{Q_0}{C}$$



- As charge consists of static and time-dependent parts

$$I = \frac{dq}{dt} \quad \text{or} \quad q = \int I dt + Q_0$$

- e-f inductance L
- no contribution to the contour integral (perfect conductor),
- Integral over the fixed surface → reasonable to neglect flux linking the open surface everywhere except the *N*-turn coil, because the magnetic field threading through in the coil is larger than everywhere else

$$-\frac{d}{dt}\iint_S \vec{B}\cdot d\vec{s} = -\frac{dN\Phi}{dt} = -\frac{d(LI)}{dt} = -L\frac{dI}{dt}$$

For solenoid: $B = \mu_0 n I$

and

$$N\Phi = NBS = \left(\mu_0 n^2 l S\right) I \equiv LI$$

Inductance

- Finally:

$$V_{ba} = IR + \frac{1}{C}\int I dt + \frac{Q_0}{C} + L\frac{dI}{dt}$$

Circuits-3

- At static condition:
- For a harmonic signal:

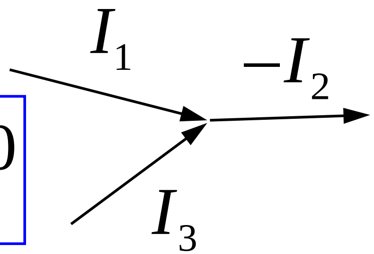
$$V_{ba}(t) = Ve^{i\omega t}$$

$$I(t) = Ie^{i\omega t}$$

$$V_{ba} = IR + \frac{1}{i\omega C} I + i\omega LI$$

- Note the phase shift between I and V
- Also conservation of charge leads to Kirchoff's current law

$$\sum_n I_n = 0$$



- Assumptions made:

1. All linear dimensions are much less than $(f\sqrt{\mu\varepsilon})^{-1}$.
2. A filamentary closed path is employed.
3. Perfect conductors exist everywhere in the circuit except at the input gap, between capacitor plates, and between the *resistor* terminals.
4. Displacement current is confined to the *capacitor*.
5. Magnetic flux is confined to the *inductor*.
6. The geometry is fixed in time.

From Neff, 1991

Overview of Electromagnetics